Optimal Rate Allocation for Shape-Gain Gaussian Quantizers

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Abstract — **We derive the optimal rate allocation of the shape and gain components of a recently proposed shape-gain quantizer for the Gaussian source [1]. The rate allocation results are of particular interest because the shape-gain quantizer addressed in this paper is the best vector quantizer known for the memoryless Gaussian source at rates of three or higher.**

I. Introduction

An important goal in source coding is to design quantizers that have both low implementation complexity and performance close to the distortion-rate function of a source. One promising new vector quantizer for the Gaussian source performs within 1 dB of the rate-distortion function for rates of 2 and higher and has an encoding complexity which is linear in the rate [1].

A *shape-gain vector quantizer* decomposes a source vector X into a *gain* $g = \|X\|$ and *shape* $S = X/g$, which are quantized to $\hat{g}$ and $\hat{S}$, respectively, and the output is $\hat{X} = \hat{g}\hat{S}$ (see Figure 1). The recently proposed wrapped spherical vector quantizer for the Gaussian source [1] contains a gain codebook that is the globally optimal scalar quantizer for the generalized Rayleigh-distributed gain $g = \|X\|$, and a shape codebook obtained by mapping a $(k-1)$-dimensional lattice Λ onto the unit k-dimensional sphere Ω_k in such a way that distance properties of Λ are nearly preserved. As a result, the distortion of the shape codebook for a source uniformly distributed on Ω_k is nearly the same as the distortion performance of Λ for a uniform source in $\mathbb{R}^{k-1}$. Furthermore, it turns out that the distortion of the overall shape-gain quantizer for the memoryless Gaussian source decomposes into a gain distortion $D_g = \frac{1}{k}E[(g-\hat{g})^2]$ which is easily numerically computed by evaluating a one-dimensional integral, and a shape distortion $D_s = \sigma^2 E[\|S-\hat{S}\|^2] \approx (k-1)\sigma^2 G(\Lambda)V(\Lambda)^{\frac{2}{k-1}}$, where σ^2 is the variance of the Gaussian source, $G(\Lambda)$ is the normalized second moment of a Voronoi region of Λ, and $V(\Lambda)$ is its volume.

II. Allocation of Shape and Gain Rates

Let R be the transmission rate of the wrapped SVQ and let the shape code rate R_s and gain code rate R_g satisfy $R_s + R_g = R$. For a numerical solution, the optimal bit allocation between shape and gain codebooks can be converged upon by evaluating $D_s + D_g$ for each rate allocation and using a gradient descent algorithm.

We can analytically determine the optimum rate allocation for asymptotically high rates. For general shape-gain quantizers this is an unsolved problem. Since the transmission rate R, the shape quantizer rate R_s, and the gain quantizer rate R_g are related by $R = R_s + R_g$ we can write the shape and gain distortions as

$$D_s \approx (k-1)\sigma^2 G(\Lambda)V(\Lambda)^{\frac{2}{k-1}} \approx C_s 2^{-2R_s\left(\frac{k}{k-1}\right)} \quad (1)$$

$$D_g \approx C_g 2^{-2R_g k} = C_g 2^{-2k(R-R_s)} \quad (2)$$

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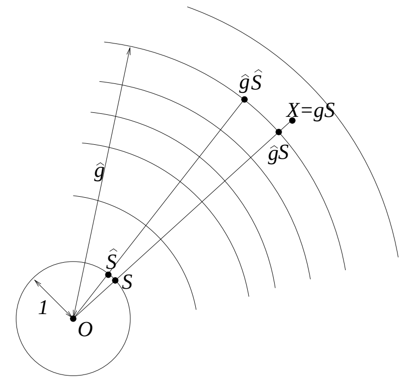

Figure 1: Geometrical view of shape-gain encoder.

where (2) holds for large R and R_s from Bennett's integral [2], and where C_s and C_g are constants that are independent of R_s and R_g. The optimal choice of R_s and R_g is given in the following theorem.

Theorem 1 *Let $X \in \mathbb{R}^k$ be an uncorrelated Gaussian vector with zero mean and component variances σ^2 and let Λ be a lattice in $\mathbb{R}^{k-1}$ with normalized second moment $G(\Lambda)$. Suppose X is quantized by a k-dimensional shape-gain vector quantizer at rate $R = R_s + R_g$ (where R_s and R_g are the shape and gain quantizer rates) with independent shape and gain encoders and whose shape codebook is a wrapped spherical code constructed from Λ. Then as $R \to \infty$, the minimum mean squared quantization error D decays as*

$$D \approx C_s \left(\frac{k}{k-1}\right)\left(\frac{C_g}{C_s}(k-1)\right)^{1/k} \cdot 2^{-2R} \quad (3)$$

and is achieved by

$$R_s = \left(\frac{k-1}{k}\right)\left[R + \frac{1}{2k}\log_2\left(\frac{C_s}{C_g}\cdot\frac{1}{k-1}\right)\right] \quad (4)$$

$$R_g = \left(\frac{1}{k}\right)\left[R - \frac{k-1}{2k}\log_2\left(\frac{C_s}{C_g}\cdot\frac{1}{k-1}\right)\right] \quad (5)$$

where $C_s = \sigma^2 \cdot (k-1)G(\Lambda)\left(\frac{2\pi^{k/2}}{\Gamma(k/2)}\right)^{\frac{2}{k-1}}$ *and* $C_g = \sigma^2 \cdot \frac{3^{k/2}\Gamma^3\left(\frac{k+2}{6}\right)}{8\Gamma(k/2)}$.

Note that for large R, the optimal allocation of transmission rate between the shape quantizer and the gain quantizer is approximately $R_s \approx \left(1-\frac{1}{k}\right)R$ and $R_g \approx \frac{1}{k}R$, as intuition would indicate. This corresponds, to within 1% when $R \geq 5$, of what was observed in the numerical rate allocation optimization.

References

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